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Duality and Lipschitzian Selections in Best Approximation from Nonconvex Cones*

VASANT A. UBHAYA

Department of Computer Science and Operations Research, Division of Mathematical Sciences, 300 Minard Hall, North Dakota State University, Fargo, North Dakota 58105

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Duality relationships in finding a best approximation from a nonconvex cone in a normed linear space in general, and in the space of bounded functions in particular, are investigated. The cone and the dual problems are defined in terms of positively homogeneous super-additive functionals on the space. Conditions are developed on the cone so that the duality gap between a pair of primal and dual problems does not exist. In addition, Lipschitz continuous selections of the metric projection are identified. The results are specialized to a convex cone. Applications are indicated to approximation problems.

1. INTRODUCTION

Duality in a normed linear space X refers to a relationship between a pair of extremum problems a primal problem on X and a dual problem on the dual space X^* of continuous linear functionals on X, or more generally, a bigger space $\hat{X}$ of nonlinear functionals on X. Given a nonconvex (i.e., not necessarily convex) cone $K \subset X$, which is defined by positively homogeneous super-additive functionals in $\hat{X}$, the primal problem is to find a best approximation to f in $X \setminus K$ from K. In this article, dual problems corresponding to this primal are defined in terms of functional in X^* and $\hat{X}$. Some basic duality results in X are established to obtain lower bounds on $d(f, K)$. When X is the space of bounded functions with weighted uniform norm, conditions are developed on K so that the "duality gap" between a pair of primal and dual problems does not exist; i.e., the optimal values of the two problems are equal. In addition, a best approximation f' to each f is identified so that the selection operator mapping f to f'

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is Lipschitzian. Some results are derived for the space of continuous functions. Examples from approximation theory illustrating the results are given.

We consider a real normed linear space X with norm $\|\cdot\|$ and its real dual X^* which is a Banach space with the norm $\|x^*\| = \sup\{|x^*(f)| : f \in X, \|f\| \leqslant 1\}$, $x^* \in X^*$. A nonempty subset K of X is called a cone (with vertex 0) if $\lambda f \in K$ whenever $f \in K$ and $\lambda \geqslant 0$. A convex cone is a cone which is convex. It is easy to verify that a cone K is convex if and only if $f + g \in K$ whenever $f, g \in K$. If K is a cone, then $d(f, K) = \inf\{\|f - g\| : g \in K\} \leqslant \|f\|$, since $0 \in K$. To motivate the discussion, we first state some basic duality results for a convex cone K. Duality in approximation has been investigated in detail in [3]. The dual, polar, or conjugate cone K^0 of K is defined by

$$K^0 = \{x^* \in X^* : x^*(f) \leqslant 0, f \in K\}.$$

It is known that K^0 is convex and weak* closed. A well known duality result is

$$d(f, K) = \max\{x^*(f)/\|x^*\| : x^* \in K^0 \setminus \{0^*\}\}, \qquad f \in X \setminus K,$$

where 0^* is the zero functional. (See, e.g., [3, Corollary 5.3(a)]; see also [8, 12, 17, 18] and other references given there.) Suppose now that $L \subset X^*$ and $L \neq \{0^*\}$. Define a convex cone K by

$$K = \{f \in X : x^*(f) \leqslant 0, x^* \in L\}.$$

Then $K^0 = \overline{\text{cc}}(L)$ and $K \neq X$, where $\overline{\text{cc}}(L)$ is the smallest weak* closed convex cone containing L. In particular, L and its convex hull $\text{co}(L)$ are in K^0 and, hence,

$$\begin{aligned} d(f, K) &\geqslant \sup\{x^*(f)/\|x^*\| : x^* \in \text{co}(L) \setminus \{0^*\}\} \\ &\geqslant \sup\{x^*(f)/\|x^*\| : x^* \in L \setminus \{0^*\}\}, \qquad f \in X \setminus K. \end{aligned} \tag{1.1}$$

When f is fixed, the first term $d(f, K) = \inf\{\|f - g\| : g \in K\}$ in (1.1) defines a primal problem on X, and the middle and the last terms define two dual problems on the space X^*. In [18], we investigated necessary and sufficient conditions so that equalities hold throughout (1.1). When equalities hold, $d(f, K)$ may be easily computed from L using the last term.

Suppose now that $L \subset \hat{X}$ is a nonempty set of real nonlinear functionals on X. We define a cone K by

$$K = \{f \in X : \hat{x}(f) \leqslant 0, \hat{x} \in L\}.$$

We assume that each $\hat{x}$ in L is a pointwise infimum of a set of functionals in X^*. This constraint is motivated by applications to approximation problems. Then each $\hat{x}$ is positively homogeneous and super-additive and, hence, concave (see Section 2). Clearly, K is not necessarily convex. In Section 2, analogous to (1.1), we develop dual problems on X^* and $\hat{X}$ which give lower bounds on $d(f, K)$. In Section 3, we apply the results to the space of bounded functions B with weighted uniform norm and obtain conditions on K so that the duality gaps do not exist. We identify Lipschitzian selection operators T mapping f to one of its best approximations f' so that $\|f' - h'\| \leqslant c\,\|f - h\|$ for some $c > 0$ and all $f, h \in B$. We also specialize the treatment to a convex cone and the space C of continuous functions. In Sections 4 and 5, we illustrate the results for nonconvex and convex cones by examples of approximation problems. In Section 5 we consider more complex cones. Our previous work on Lipschitzian selections [22, 23] required that K be closed under translation by constant functions and that the uniform norm have unity weight function. Such constraints are not required in this article. In particular, the convex cone of sub-additive functions in Example 4.3 is not closed under translation by constant functions. For additional work on continuous and Lipschitz continuous selections in approximation see [4, 5, 6, 10, 13].

2. Duality in Normed Linear Spaces

In this section we derive lower bounds on $d(f, K)$.

A real-valued nonlinear functional $\hat{x}$ on X is said to be positively homogeneous if $\hat{x}(\lambda f) = \lambda \hat{x}(f)$ for all f in X and all $\lambda \geqslant 0$. Hence $\hat{x}(0) = 0$. Let $\|\hat{x}\| = \sup\{|\hat{x}(f)| : f \in X, \|f\| \leqslant 1\}$. By positive homogeneity we have $|\hat{x}(f)| \leqslant \|\hat{x}\|\,\|f\|$ for all f in X. We say $\hat{x}$ is bounded if $\|\hat{x}\| < \infty$. We call $\hat{x}$ super-additive if $\hat{x}(f+g) \geqslant \hat{x}(f) + \hat{x}(g)$ for all f, g in X.

Proposition 2.1. *Suppose that $\hat{x}$ is a positively homogeneous super-additive functional on X.*

(a) *The following holds for all f, g in X and all $0 \leqslant \lambda \leqslant 1$.*

(i) $\hat{x}(\lambda f + (1-\lambda) g) \geqslant \lambda \hat{x}(f) + (1-\lambda)\hat{x}(g)$; *i.e., $\hat{x}$ is concave on X.*

(ii) $\hat{x}(-f) \leqslant -\hat{x}(f)$.

(iii) $|\hat{x}(f) - \hat{x}(g)| \leqslant \max\{-\hat{x}(f-g), -\hat{x}(g-f)\} \leqslant \|\hat{x}\|\,\|f-g\|$.

(b) *The following three conditions are equivalent.*

(i) *$\hat{x}$ is continuous at 0.*

(ii) *$\hat{x}$ is continuous everywhere.*

(iii) *$\hat{x}$ is bounded.*

Proof. To prove (a), we observe that (i) follows by positive homogeneity and super-additivity. Again, by super-additivity, we have $0=\hat{x}(0)\geqslant\hat{x}(f)+\hat{x}(-f)$, which gives (ii). Writing $f=(f-g)+g$, we obtain $\hat{x}(f)\geqslant\hat{x}(f-g)+\hat{x}(g)$. Interchanging f and g we obtain (iii). To show (b) we note that by positive homogeneity, if $\hat{x}$ is continuous at 0, then it is bounded. The rest of (b) follows from (a)(iii). The proof is complete.

We remark that if $\hat{X}$ denotes the set of all positively homogeneous bounded functionals on X, then $\hat{X}$ is a linear subspace with norm $\|\hat{x}\|$, and $X^*\subset\hat{X}$.

Let P and Q_p, $p\in P$, be index sets. For each p in P, let $\{x^*_{p,q}: q\in Q_p\}$ be a set of nonzero functionals in X^*. Define the pointwise infimum of $\{x^*_{p,q}: q\}$ by

$$\hat{x}_p=\inf\{x^*_{p,q}: q\in Q_p\}, \qquad p\in P, \tag{2.1}$$

and

$$K=\{k\in X: \hat{x}_p(k)\leqslant 0,\ p\in P\}. \tag{2.2}$$

Clearly, K is a cone which is not necessarily convex. We then have the following.

PROPOSITION 2.2. (a) *For each p in P, $\hat{x}_p$ is positively homogeneous and super-additive with*

$$\|\hat{x}_p\|=\sup\{\|x^*_{p,q}\|: q\in Q_p\}, \qquad p\in P. \tag{2.3}$$

All properties stated in Proposition 2.1 *apply to $\hat{x}_p$.*

(b) *If $K_{p,q}=\{k\in X: x^*_{p,q}(k)\leqslant 0\}$, then $K_{p,q}$ is a convex cone for all p, q. If Q_p is finite for all $p\in P$, then $K=\bigcap\{\bigcup\{K_{p,q}: q\in Q_p\}: p\in P\}$, $\|\hat{x}_p\|<\infty$ for all $p\in P$ and K is closed.*

Proof. (a) Clearly, $\hat{x}_p$ is positively homogeneous. To show super-additivity, let $p\in P$ and $f, g\in X$. Then $x^*_{p,q}(f+g)=x^*_{p,q}(f)+x^*_{p,q}(g)\geqslant\hat{x}_p(f)+\hat{x}_p(g)$. Hence $\hat{x}_p(f+g)\geqslant\hat{x}_p(f)+\hat{x}_p(g)$.

To establish (2.3), let $\varepsilon>0$, $p\in P$, and $c=\sup_q\{\|x^*_{p,q}\|\}\leqslant\infty$. If $0<\mu<c$, then there exists $q\in Q_p$ such that $\|x^*_{p,q}\|>\mu$. Again, there exists g in X with $\|g\|=1$ such that $x^*_{p,q}(g)\leqslant-\|x^*_{p,q}\|+\varepsilon$. Hence, $\hat{x}_p(g)\leqslant-\|x^*_{p,q}\|+\varepsilon$. It follows that $\|\hat{x}_p\|\geqslant-\hat{x}_p(g)\geqslant\|x^*_{p,q}\|-\varepsilon>\mu-\varepsilon$, which gives $\|\hat{x}_p\|\geqslant c$. If $c=\infty$ then (2.3) is shown to hold. Otherwise, let $f\in X$ with $\|f\|=1$.

Then $\hat{x}_p(f) \leqslant x^*_{p,q}(f) \leqslant \|x^*_{p,q}\| \leqslant c$. Also there exists $q \in Q_p$ such that $x^*_{p,q}(f) - \varepsilon \leqslant \hat{x}_p(f)$. Hence,

$$-\hat{x}_p(f) \leqslant -x^*_{p,q}(f) + \varepsilon \leqslant \|x^*_{p,q}\| + \varepsilon \leqslant c + \varepsilon.$$

We conclude that $|\hat{x}_p(f)| \leqslant c + \varepsilon$ and $\|\hat{x}_p\| \leqslant c$. Thus (2.3) holds.

(b) If Q_p is finite then, by (2.3), $\|\hat{x}_p\| < \infty$. By Proposition 2.1(b), $\hat{x}_p$ is continuous for each p and, hence, K is closed. The remaining assertion about K follows immediately.

The proof is complete.

PROPOSITION 2.3. *Let K_i, $i \in I$, be an arbitrary collection of nonempty subsets of X and $f \in X$. Then the following holds.*

(a) $d(f, \bigcup_i K_i) = \inf\{d(f, K_i) : i \in I\}$,

(b) $d(f, \bigcap_i K_i) \geqslant \sup\{d(f, K_i) : i \in I\}$.

Proof. To prove (a), denote its right-hand side by ρ, let $K' = \bigcup K_i$, and let $\varepsilon > 0$. Then there exists some $j \in I$ such that $d(f, K_j) < \rho + \varepsilon/2$. Again there exists $k \in K_j$ such that $\|f - k\| < d(f, K_j) + \varepsilon/2 < \rho + \varepsilon$. Since $k \in K'$ we have $d(f, K') \leqslant \|f - k\|$, which gives $d(f, K') \leqslant \rho$. Now, if $k \in K'$ then $k \in K_i$ for some i, and $\|f - k\| \geqslant d(f, K_i) \geqslant \rho$. Hence $d(f, K') \geqslant \rho$ and (a) is established. The proof for (b) is simpler. The proof is complete.

THEOREM 2.1 (Duality bounds for nonconvex cone K). *Assume Q_p is finite for each $p \in P$. Let $f \in X$ and define*

$$\alpha(f) = \sup\{\inf\{x^*_{p,q}(f)/\|x^*_{p,q}\| : q \in Q_p\} : p \in P\},$$

$$\beta(f) = \sup\{\hat{x}_p(f)/\|\hat{x}_p\| : p \in P\}.$$

Then $d(f, K) \geqslant \alpha(f) \geqslant \beta(f)$ for $f \in X \setminus K$. If for each $p \in P$, $\|\hat{x}_p\| = \|x^*_{p,q}\|$ for all $q \in Q_p$, then $d(f, K) \geqslant \alpha(f) = \beta(f)$ for $f \in X \setminus K$.

We first establish the following lemma.

LEMMA 2.1. *Assume Q_p is finite for each $p \in P$. Then the following are equivalent.*

(a) $f \in X \setminus K$.

(b) $\alpha(f) > 0$.

(c) $\beta(f) > 0$.

Proof. By Proposition 2.2(b), $\|\hat{x}_p\| < \infty$ for all $p \in P$. Hence, by the definition of K, (a) and (c) are equivalent. Now (b) holds if and only if $x^*_{p,q}(f) > 0$ for some $p \in P$ and all $q \in Q_p$ which is equivalent to (c). The proof is complete.

Proof of Theorem 2.1. We first establish that $\alpha(f) \geqslant \beta(f)$, $f \in X \backslash K$. By Lemma 2.1, it suffices to consider $p \in P$ with $\hat{x}_p(f) > 0$. For such p, if $q \in Q_p$, we have $x^*_{p,q}(f) \geqslant \hat{x}_p(f) > 0$. Since, by (2.3), $\|x^*_{p,q}\| \leqslant \|\hat{x}_p\| < \infty$, we find that $x^*_{p,q}(f)/\|x^*_{p,q}\| \geqslant \hat{x}_p(f)/\|\hat{x}_p\|$, which establishes $\alpha(f) \geqslant \beta(f)$.

To show $d(f, K) \geqslant \alpha(f)$, let $k \in K_{p,q}$ which is defined in Proposition 2.2(b). Then $x^*_{p,q}(k) \leqslant 0$. Hence, $\|x^*_{p,q}\| \, \|f-k\| \geqslant x^*_{p,q}(f-k) \geqslant x^*_{p,q}(f)$, which gives $\|f-k\| \geqslant x^*_{p,q}(f)/\|x^*_{p,q}\| = c_{p,q}$, say. Thus, $d(f, K_{p,q}) \geqslant c_{p,q}$. (This inequality also follows from (1.1) by letting L be the singleton set $\{x^*_{p,q}\}$.) We now use Proposition 2.3. If $K_p = \bigcup \{K_{p,q} : q \in Q_p\}$ then we have $d(f, K_p) = \min_q \{d(f, K_{p,q})\} \geqslant \min_q \{c_{p,q}\}$. Since $K = \bigcap \{K_p : p \in P\}$, we find that $d(f, K) \geqslant \sup_p \{d(f, K_p)\} = \alpha(f)$.

Finally to show $\alpha(f) = \beta(f)$ under the stated condition $\|\hat{x}_p\| = \|x^*_{p,q}\|$, we obtain $\hat{x}_p(f)/\|\hat{x}_p\| = \inf\{x^*_{p,q}(f)/\|x^*_{p,q}\| : q \in Q_p\}$ for each p from the definition of $\hat{x}_p$. It follows that $\alpha(f) = \beta(f)$. The proof is complete.

3. Duality and Lipschitzian Selections in Uniform Approximation

In this section, we consider the problem of uniform approximation and obtain conditions under which $d(f, K) = \alpha(f)$ for $f \in X \backslash K$ in Theorem 2.1. We also identify Lipschitzian selections as defined in Section 1. We consider two cases when K is a nonconvex cone and a convex cone. In later sections we apply the results to problems in approximation theory. The following example will show that, in general, $d(f, K) > \beta(f)$ for $f \in X \backslash K$, however, under certain conditions we will establish that $d(f, K) = \alpha(f) = \beta(f)$. Let X be the real line and $x^*_{11}(f) = 2f$, $x^*_{12}(f) = f$, where $f \in X$, and $\hat{x} = \min\{x^*_{11}, x^*_{12}\}$. Then $K = (-\infty, 0]$, $\|\hat{x}\| = \|x^*_{11}\| = 2$ and $\|x^*_{12}\| = 1$. If $f = 1$, then $1 = d(f, K) = \alpha(f) > \beta(f) = \frac{1}{2}$.

Let S be any set and $0 < w(s) < \infty$ for all s in S be a weight function on S. Let B denote the set of all real functions f on S such that $\|f\| = \|f\|_w = \sup\{w(s) \, |f(s)| : s \in S\} < \infty$. Then B is a Banach space with norm $\|\cdot\|$, which is called the weighted uniform norm. Note that an f in B is not necessarily bounded on S. Let $\mathscr{A}$ be the set consisting of certain nonempty countable subsets of S. Suppose there is a mapping $\tau : \mathscr{A} \to 2^S \backslash \{\varnothing\}$ such that $\tau(A) \cap A = \varnothing$ for all $A \in \mathscr{A}$. Let S' denote $\bigcup \{\tau(A) : A \in \mathscr{A}\}$, i.e., all the elements of S in the range of τ. For A in $\mathscr{A}$, let $q = q_A$ denote a nonnegative function on A such that $|q| = \sum \{q(t)/w(t) : t \in A\} < \infty$. Let $P = \{(A, s) : A \in \mathscr{A}, \, s \in \tau(A)\}$ and Q_p, for each $p = (A, s) \in P$, be a set of above defined functions q on A with $|q| < \infty$. We define linear functionals indexed by $p = (A, s) \in P$ and $q \in Q_p$ as follows.

$$x^*_{p,q}(f) = f(s) - \sum \{q(t) f(t) : t \in A\}. \tag{3.1}$$

We use the notation of Section 2 with $X = B$ and define $\hat{x}_p$ and K by (2.1) and (2.2), respectively. Clearly, $\hat{x}_p$ may be written as

$$\hat{x}_p(f) = f(s) - \sup\left\{\sum_{t \in A} q(t) f(t) : q \in Q_p\right\}. \tag{3.2}$$

LEMMA 3.1. *Suppose that $G \subset K$ is nonempty, and for all g in G, $g \leqslant f$ holds for some f in B. Let $k(s) = \sup\{g(s) : g \in G\}$, $s \in S$. Then $k \in K$.*

Proof. Clearly, $k \in B$. Now, for all g in G we have $g \leqslant k$ and $\hat{x}_p(g) \leqslant 0$ for all $p \in P$. Suppose $s \in S'$. Then, for any $p = (A, s) \in P$, $q \in Q_p$, and $g \in G$, it follows from (3.2) by the nonnegativity of q that

$$g(s) \leqslant \sup_q \left\{\sum_A q(t) g(t)\right\} \leqslant \sup_q \left\{\sum_A q(t) k(t)\right\}.$$

Hence $k(s) \leqslant \sup_q \{\sum q(t) k(t)\}$ which is $\hat{x}_p(k) \leqslant 0$. Thus $k \in K$. The proof is complete.

Next we establish the existence of a best approximation when Q_p is finite for each p. A best approxmation f' of f is called the maximal best approximation if $f' \geqslant g$ for all best approximations g to f.

PROPOSITION 3.1. *Suppose that Q_p is finite for each p. Then, every f in B has a maximal best approximation from K.*

Proof. For convenience let $d(f, K) = \rho$, $u = f - \rho/w$, and $v = f + (\rho + 1)/w$. Clearly, $u, v \in B$. Now for each n, there exists $f_n \in K$ such that $\|f - f_n\| \leqslant \rho + 1/n = \rho_n$, say. Then $f - \rho_n/w \leqslant f_n \leqslant f + \rho_n/w \leqslant v$ since $\rho_n \leqslant \rho + 1$. Define $g_n = \sup\{f_m : m \geqslant n\}$. Then $g_n \geqslant g_{n+1}$, and, by Lemma 3.1, $g_n \in K$. Since $g_n \geqslant f_m$ for all $m \geqslant n$ and $\rho_{n+1} \leqslant \rho_n$, we have $f - \rho_m/w \leqslant f_m \leqslant g_n \leqslant f + \rho_n/w$ for all $m \geqslant n$. Letting $m \to \infty$, we obtain

$$u = f - \rho/w \leqslant g_n \leqslant f + \rho_n/w \leqslant v,$$

for all n. If $g(s) = \lim g_n(s)$, $s \in S$, then we conclude that $f - \rho/w \leqslant g \leqslant f + \rho/w$ which is $\|f - g\| \leqslant \rho$. We show that $g \in K$; this will establish that g is a best approximation. As shown above, we have $u \leqslant g_n \leqslant v$. Now $\sum_A q(t)\, |u(t)| \leqslant \|u\|\, |q| < \infty$ and $\sum_A q(t)\, |v(t)| \leqslant \|v\|\, |q| < \infty$. Hence, by the bounded convergence theorem [7], $\sum_A q(t)\, g_n(t) \to \sum_A q(t)\, g(t)$ as $n \to \infty$. We conclude that $x^*_{p,q}(g_n) \to x^*_{p,q}(g)$ for all $q \in Q_p$. Since Q_p is finite we have $\hat{x}_p(g_n) \to \hat{x}_p(g)$. Again since $g_n \in K$ we have $\hat{x}_p(g_n) \leqslant 0$ and, hence, $\hat{x}_p(g) \leqslant 0$ for all p. Thus $g \in K$ and is a best approximation. Now if G is the set of all best approximations, then $g \leqslant f + \rho/w$ for all g in G. Since $f + \rho/w \in B$, by Lemma 3.1, $f'(s) = \sup\{g(s) : g \in G\}$ is in K. It is easy to verify that f' is a best approximation. Clearly, it is the maximal best approximation. The proof is complete.

PROPOSITION 3.2. *Let* $p = (A, s)$ *and* $q \in Q_p$.

(a) $\|x^*_{p,q}\| = 1/w(s) + \sum\{q(t)/w(t) : t \in A\} = 1/w(s) + |q|$.

(b) $\|\hat{x}_p\| = 1/w(s) + \sup\{|q| : q \in Q_p\} = \sup\{\|x^*_{p,q}\| : q \in Q_p\}$.

Proof. (a) If $f \in B$ and $\|f\| \leqslant 1$, then $|f(u)| \leqslant 1/w(u)$ for all u in S. Hence,

$$|x^*_{p,q}(f)| \leqslant |f(s)| + \sum q(t)\,|f(t)| \leqslant 1/w(s) + \sum q(t)/w(t) = 1/w(s) + |q|.$$

Now, define g on S by $g(s) = -1/w(s)$, $g(t) = 1/w(t)$ for t in A, and 0 elsewhere. Since $s \in S \backslash A$, this is possible. Clearly, $\|g\| = 1$. Then $x^*_{p,q}(g) = 1/w(s) + |q|$ and the result follows.

(b) This follows at once from (a) and (2.3).

The proof is complete.

For each f in B, let $K_f = \{k \in K : k \leqslant f\}$ and $\bar{f}(s) = \sup\{k(s) : k \in K_f\}$, $s \in S$. If $\bar{f} \in K$, then $\bar{f}$ is called the greatest K-minorant of f. Similarly, letting $K'_f = \{k \in K : k \geqslant f\}$, define $\underline{f}(s) = \inf\{k(s) : k \in K'_f\}$, $s \in S$. If $\underline{f} \in K$, it is called the smallest K-majorant of f.

PROPOSITION 3.3. *Suppose* $f \in B$. *Then the following* (a)–(c) *are equivalent and imply* (d).

(a) $K_f \neq \varnothing$.

(b) $\bar{f} \in K$.

(c) $\bar{f}(s) > -\infty$ *for all* s *in* S.

(d) $\bar{f} \leqslant f$ *and* $\bar{f}(s) = f(s)$ *if* $s \in S \backslash S'$. *Hence, if* $S_f = \{s \in S : f(s) > \bar{f}(s)\}$, *then* $S_f \subset S'$.

Proof. If (a) holds then by Lemma 3.1 with $G = K_f$ we have that $\bar{f} \in K$, which is (b). If (b) holds, then $\bar{f} \in B$ and (c) holds. If $K_f = \varnothing$, then $\bar{f} = -\infty$. Hence (c) implies (a). If (b) holds then define g on S by $g(s) = \bar{f}(s)$ for $s \in S'$, and $g(s) = f(s)$, otherwise. Then g is in B and satisfies

$\hat{x}_p(g) \leqslant 0$ for all p. Thus $g \in K$, and, consequently, $g \leqslant \bar{f}$. This gives $\bar{f}(s) = f(s)$ for $s \in S \setminus S'$ which is (d). The proof is complete.

We let $P_s = \{p = (A, s): p \in P\}$, $s \in S'$. One may easily verify that $P_s \neq \varnothing$ for all s in S' and $P = \bigcup \{P_s: s \in S'\}$. For $h \in B$ define h^0 by

$$\begin{aligned} h^0(s) &= \min\{h(s), h(s) - \sup\{\hat{x}_p(h): p \in P_s\}\}, \qquad s \in S', \\ &= h(s), \qquad s \in S \setminus S'. \end{aligned} \tag{3.3}$$

Substituting for $\hat{x}_p$ from (3.2) in (3.3) we obtain

$$h^0(s) = \min\left\{h(s), \inf\left\{\sup\left\{\sum_A q(t)\, h(t): q \in Q_p\right\}: p \in P_s\right\}\right\}, \qquad s \in S'. \tag{3.4}$$

Note that if k, $h \in B$ and $k \leqslant h$, then (3.4) shows that $k^0 \leqslant h^0$. Moreover, if $k \in K$, then $\hat{x}_p(k) \leqslant 0$ for all p, and hence by (3.3), we have $k^0 = k$. For each f in $B \setminus K$, define $h_f = f + \alpha(f)/w$, where $\alpha(f)$ is defined in Theorem 2.1. Then $\|h_f\| \leqslant \|f\| + \alpha(f) \leqslant \|f\| + d(f, K)$ and $h_f \in B$. This h_f will play an important role in the following analysis. Letting $h = h_f$ for convenience, we define $h^0 = h_f^0$ by (3.3).

Proposition 3.4. *Assume that Q_p is finite for each $p \in P$. If $f \in B \setminus K$ then $h^0 = h_f^0 \in B$ with $\|h - h^0\| \leqslant 2\alpha(f)$. If $h^0 \in K$, then $h^0 = \bar{h}$, where $\bar{h}$ is the greatest K-minorant of $h = h_f$. Consequently, $\|h - \bar{h}\| \leqslant 2\alpha(f)$ and*

$$\begin{aligned} \bar{h}(s) &= \min\left\{h(s), \inf\left\{\max\left\{\sum_A q(t)\, h(t): q \in Q_p\right\}: p \in P_s\right\}\right\}, \qquad s \in S' \\ &= h(s), \qquad s \in S \setminus S'. \end{aligned} \tag{3.5}$$

Corollary. *If $h = f + \rho/w$, where $\alpha(f) \leqslant \rho < \infty$ and h^0 is defined by (3.3) for this h, then the above proposition holds with $\alpha(f)$, h_f, and h_f^0 there replaced respectively by ρ, h, and h^0.*

Proof. Since Q_p is finite, by Proposition 2.2(b), we have $\|\hat{x}_p\| < \infty$. For convenience, let $\theta = \alpha(f)$. Also let $s \in S'$ and $p \in P_s$. Then, by the definition of θ, we have $\theta \geqslant \min\{x^*_{p,q}(f)/\|x^*_{p,q}\|: q \in Q_p\}$. Hence, there exists $q \in Q_p$ such that $\theta \geqslant x^*_{p,q}(f)/\|x^*_{p,q}\|$, which gives $\theta\, \|x^*_{p,q}\| \geqslant x^*_{p,q}(f)$. By substituting $f = h - \theta/w$ we obtain

$$\theta(\|x^*_{p,q}\| + x^*_{p,q}(1/w)) \geqslant x^*_{p,q}(h).$$

Again, $\|x^*_{p,q}\| = 1/w(s) + |q|$ by Proposition 3.2(a), and $x^*_{p,q}(1/w) = 1/w(s) - |q|$ as may be easily verified. Hence $2\theta/w(s) \geqslant x^*_{p,q}(h)$, which gives $2\theta/w(s) \geqslant \hat{x}_p(h)$ for all $p \in P_s$. Then by (3.3) we have $h(s) \geqslant h^0(s) \geqslant h(s) - 2\theta/w(s)$. Also, $h^0(s) = h(s)$ if $s \in S \setminus S'$. Thus $h^0 \in B$ with $\|h - h^0\| \leqslant 2\theta$.

We now show that if $k \in K$ and $k \leqslant h$ then $k \leqslant h^0$. Since, by assumption $h^0 \in K$, this will establish that $h^0 = \bar{h}$. Let $s \in S'$ and $p = (A, s) \in P_s$. Then, by (3.2), there exists $q \in Q_p$ such that

$$\hat{x}_p(k) = k(s) - \sum_A q(t)\, k(t).$$

Also,

$$\hat{x}_p(h) \leqslant h(s) - \sum_A q(t)\, h(t).$$

Hence,

$$\hat{x}_p(h) - \hat{x}_p(k) \leqslant (h-k)(s) - \sum_A q(t)(h(t) - k(t)).$$

Since $k \leqslant h$ and $\hat{x}_p(k) \leqslant 0$ as $k \in K$, we obtain from the above inequality that $k(s) \leqslant h(s) - \hat{x}_p(h)$ for all $p \in P_s$. Now, since $k \leqslant h$, we conclude from (3.3) that $k(s) \leqslant h^0(s)$, $s \in S'$. For $s \in S \backslash S'$, we have $k(s) \leqslant h(s) = h^0(s)$. Hence $k \leqslant h^0$ and $h^0 = \bar{h}$ as asserted. From (3.4) we obtain (3.5). The corollary may be proved exactly as above. The proof is complete.

The above proposition is fundamental in establishing our next theorem.

THEOREM 3.1 (Duality for nonconvex K). *Assume that Q_p is finite for each $p \in P$ and $h^0 = h_f^0 \in K$ for each $f \in B \backslash K$. Then*

$$d(f, K) = \alpha(f) = \sup\{\min\{x^*_{p,q}(f)/\|x^*_{p,q}\| : q \in Q_p\} : p \in P\}, \qquad f \in B\backslash K. \tag{3.6}$$

Furthermore, $f' = \bar{h}$ is the maximal best approximation to f with $\|h - \bar{h}\| \leqslant 2\alpha(f)$.

COROLLARY. *Under the hypothesis of the theorem, if for each p in P, $\|\hat{x}_p\| = \|x^*_{p,q}\|$ holds for all $q \in Q_p$, then*

$$d(f, K) = \alpha(f) = \beta(f), \qquad f \in X\backslash K.$$

Proof. Let $\theta = \alpha(f)$ for convenience. By Proposition 3.4 we have $h^0 = \bar{h}$ and $\|h - \bar{h}\| \leqslant 2\theta$. This gives $h - \bar{h} \leqslant 2\theta/w$. Now since $f = h - \theta/w$, we obtain $f - \bar{h} = h - \bar{h} - \theta/w \leqslant \theta/w$. Again, since $h \geqslant \bar{h}$, we have $f - \bar{h} \geqslant -\theta/w$. Thus $\|f - \bar{h}\| \leqslant \theta$. Now, $\bar{h} \in K$, and by Theorem 2.1, $d(f, K) \geqslant \theta$. We conclude that $d(f, K) = \|f - \bar{h}\| = \theta$.

If g is any best approximation to f then $\|f - g\| = \theta$ and hence, $f - \theta/w \leqslant g \leqslant f + \theta/w$. Since $g \in K$ and $\bar{h}$ is the greatest K-minorant of h, we have $g \leqslant \bar{h}$ and $\bar{h}$ is the maximal best approximation.

The corollary follows immediately since, by Theorem 2.1, $\alpha(f)=\beta(f)$ under the condition on the norms. The proof is complete.

We remark that the condition, $h^0=h_f^0\in K$ for each $f\in B\backslash K$, of the above theorem may be replaced by the stronger condition, $h^0\in K$ for each $h\in B$. Note that (3.6) allows us to compute $d(f,K)$ easily using the defining functionals $x^*_{p,q}$ of the cone K.

THEOREM 3.2 (Alternative forms of duality for nonconvex K). *Suppose that the hypothesis of Theorem* 3.1 *holds. Let $f\in B\backslash K$ and*

$$\hat{y}_p=\min\{x^*_{p,q}/\|x^*_{p,q}\|:q\in Q_p\},\qquad p\in P,$$
$$\bar{f}(s)=\sup\{\hat{y}_p(f):p\in P_s\},\qquad s\in S'.$$

Then $\hat{y}_p$ is a positively homogeneous super-additive functional on B with $\|\hat{y}_p\|=1$ and the following duality holds:

$$d(f,K)=\sup\{\hat{y}_p(f):p\in P\}=\sup\{\bar{f}(s):s\in S'\},\qquad f\in B\backslash K.$$

Proof. Define $y^*_{p,q}=x^*_{p,q}/\|x^*_{p,q}\|$. Then $\|y^*_{p,q}\|=1$ for all p,q and $\hat{y}_p=\min\{y^*_{p,q}:q\}$. As in Proposition 2.2, $\hat{y}_p$ is positively homogeneous super-additive and by (2.3), $\|\hat{y}_p\|=1$. The duality is simply a restatement of Theorem 3.1. The proof is complete.

We now investigate Lipschitzian selections. For each $p=(A,s)\in P$ and $q\in Q_p$, let $\sigma_{p,q}=\sum\{q(t):t\in A\}$. Define

$$\sigma=\sup\{\sigma_{p,q}:p\in P,q\in Q_p\}.$$

THEOREM 3.3 (Lipschitzian selections for nonconvex K). *Suppose that the hypothesis of Theorem* 3.1 *holds and $\sigma<\infty$. Let f' be the maximal best approximation to f in B. (If $f\in K$ then $f'=f$.) Then the selection operator $T:B\to K$, defined by $T(f)=f'$, is Lipschitzian satisfying $\|T(f)-T(g)\|\leqslant c\,\|f-g\|$ for all $f,g\in B$, where $c=2\max\{1,\sigma\}$.*

Proof. First assume that $f,g\in B\backslash K$. Let $\varepsilon>0$, $h=f+\alpha(f)/w$, and $k=g+\alpha(g)/w$. Then by Theorem 3.1 we have $f'=\bar{h}$ and $g'=\bar{k}$. Let $s\in S_k$, where S_k is as defined in Proposition 3.3(d), and $\varepsilon'=\varepsilon/w(s)$. By (3.5) we have

$$\bar{k}(s)=\inf\left\{\max\left\{\sum_A q(t)\,k(t):q\in Q_p\right\}:p\in P_s\right\}.$$

Hence, there exists $p\in P_s$ such that

$$\bar{k}(s)\geqslant\max\left\{\sum_A q(t)\,k(t):q\in Q_p\right\}-\varepsilon'. \tag{3.7}$$

Then, by (3.5) we have

$$\bar{h}(s) \leqslant \max\left\{\sum_A q(t)\, h(t) : q \in Q_p\right\}. \tag{3.8}$$

Again, there exists $q \in Q_p$ such that $\bar{h}(s) \leqslant \sum q(t)\, h(t)$. Then $\bar{k}(s) \geqslant \sum q(t)\, k(t) - \varepsilon'$. We obtain, therefore, $\bar{h}(s) - \bar{k}(s) \leqslant \sum q(t)(h(t) - k(t)) + \varepsilon'$. Multiplying both sides by $w(s)$ we have

$$w(s)(\bar{h}(s) - \bar{k}(s)) \leqslant \sigma_{p,q} \|h - k\| + \varepsilon \leqslant \sigma \|h - k\| + \varepsilon,$$

for all $s \in S_k$. If $s \in S \backslash S_k$, then by Proposition 3.3(d), we have $\bar{k}(s) = k(s)$. Since $\bar{h}(s) \leqslant h(s)$ we obtain $w(s)(\bar{h}(s) - \bar{k}(s)) \leqslant \|h - k\|$. We then have $w(s)(\bar{h}(s) - \bar{k}(s)) \leqslant c' \|h - k\| + \varepsilon$ for all s in S where $c' = c/2$. Interchanging h and k we obtain $\|\bar{h} - \bar{k}\| \leqslant c' \|h - k\|$. By Theorem 3.1, $\alpha(f) = d(f, K)$ and $\alpha(g) = d(g, K)$, and also $|d(f, K) - d(g, K)| \leqslant \|f - g\|$, by a well known result. We conclude that

$$\|h - k\| \leqslant \|f - g\| + |\alpha(f) - \alpha(g)| \leqslant 2 \|f - g\|,$$

whence we have $\|\bar{h} - \bar{k}\| \leqslant c \|f - g\|$.

Now suppose that $f \in K$ and $g \in B \backslash K$. Then we let $k = g + \alpha(g)$ as before and $\bar{h} = h = f$ (i.e., consider $\alpha(f) = 0$). Suppose $s \in S_k$. Then (3.7) holds as before. Since $f \in K$, we find that $\hat{x}_p(f) \leqslant 0$ for $p = (A, s)$, where s and A are as in (3.7). This is equivalent to (3.8) with $\bar{h} = h = f$. The rest of the argument may be carried out as above to show that $w(s)(\bar{h}(s) - \bar{k}(s)) \leqslant \sigma \|h - k\|$. Now $\bar{k}(s) \leqslant k(s)$ and $\bar{h}(s) = h(s) = f(s)$. Hence, $w(s)(\bar{k}(s) - \bar{h}(s)) \leqslant \|h - k\|$. Thus, $w(s)\, |\bar{h}(s) - \bar{k}(s)| \leqslant c' \|h - k\|$ for all $s \in S_k$. If $s \in S \backslash S_k$, then we argue as above, and noting that $\alpha(g) \leqslant \|f - g\|$, we complete the proof of $\|\bar{h} - \bar{k}\| \leqslant c \|f - g\|$. If $f, g \in K$ then the result holds. The proof is complete.

We now consider special cases of K. Define $\mu(w) = \sup\{w(s) : s \in S\}$ and $\lambda(w) = \inf\{w(s) : s \in S\}$.

LEMMA 3.2. *Assume that* $\sigma_{p,q} = 1$ *for all* $p \in P$ *and* $q \in Q_p$.

(a) *If* $\mu(w) < \infty$, *then all constant functions are in* K *and* K *is closed under translation by these functions.*

(b) *If* $0 < \lambda(w) \leqslant \mu(w) < \infty$, *then* K *has properties as stated in* (a), $\bar{f} \in K$ *for all* f *in* B, *and* $\|\hat{x}_p\| \leqslant 2/\lambda(w) < \infty$ *for all* p *in* P.

(c) *If* $w = 1$, *then conclusions of* (a) *and* (b) *hold, and* $\|\hat{x}_p\| = \|x^*_{p,q}\| = 2$ *for all* $p \in P$ *and* $q \in Q_p$.

Proof. (a) The condition $\mu(w) < \infty$ shows that all constant functions are in B. Since $\sigma_{p,q} = 1$, substitution in (3.2) verifies that $\hat{x}_p(f+\alpha) = \hat{x}_p(f)$ for all real α and $f \in B$. Hence, K is translation-invariant as stated. Since $0 \in K$, all constant functions are in K.

(b) We have $f \geqslant -\|f\|/w \geqslant -\|f\|/\lambda(w) = \rho$, say. Since $\mu(w) < \infty$, by (a) we have that $\rho \in K$ and hence $K_f \neq \emptyset$. By Proposition 3.3, $\bar{f} \in K$. Also, by Proposition 3.2, $\|\hat{x}_p\| \leqslant 2/\lambda(w)$ for all p.

(c) This follows from Proposition 3.2.

The proof is complete.

Next we apply Theorems 3.1–3.3 to nonconvex K under special conditions.

PROPOSITION 3.5. *Suppose that the hypothesis of Theorem* 3.1 *holds. Also assume that* $\mu(w) < \infty$ *and* $\sigma_{p,q} = 1$ *for all* $p \in P$ *and* $q \in Q_p$. *Then the conclusions of Theorems* 3.1–3.3 *hold with* $\|h - \bar{h}\| = 2\alpha(f)$ *and* $c = 2$.

Proof. By Theorem 3.1, $\|f - \bar{h}\| = \alpha(f) = \theta$, say. By Lemma 3.2(a), K is translation-invariant as stated there. Hence, given $\varepsilon > 0$, there exists $s \in S$ with $w(s)(f(s) - \bar{h}(s)) > \theta - \varepsilon$. Now $h = f + \theta/w$ and hence

$$w(s)(h(s) - \bar{h}(s)) = w(s)(f(s) - \bar{h}(s)) + \theta > 2\theta - \varepsilon.$$

Hence, $\|h - \bar{h}\| = 2\alpha(f)$. Clearly, $\sigma = 1$ and, hence, $c = 2$. The proof is complete.

Recall that $\hat{y}_p$ and $\tilde{f}(s)$ are defined in Theorem 3.2.

PROPOSITION 3.6. *Suppose that the hypothesis of Theorem* 3.1 *holds. Suppose also that* $w = 1$ *and* $\sigma_{p,q} = 1$ *for all* $p \in P$ *and* $q \in Q_p$. *Then,*

$$\begin{aligned} d(f, K) &= \alpha(f) = \beta(f) = \|f - \bar{f}\|/2 \\ &= \sup\left\{f(s) - \max\left\{\sum_A q(t)\, f(t) : q \in Q_p\right\}\right\}\Big/ 2, \qquad f \in B \setminus K, \qquad (3.9) \end{aligned}$$

where $\bar{f}$ *is the greatest K-minorant of* f *and the supremum is taken over all* $p = (A, s) \in P$. *Also,* $f(s) - \bar{f}(s) = 2\max\{\tilde{f}(s), 0\}$ *for all* $s \in S'$.

Proof. Since $w = 1$ we have $\lambda(w) = \mu(w) = 1$. Then, Lemma 3.2(c) applies showing that K is closed under translation by constant functions, $\bar{f} \in K$ for all f in B, and $\|\hat{x}_p\| = \|x^*_{p,q}\| = 2$ for all p, q. Since $h = f + \alpha(f)$, by translation-invariance we have $\bar{h} = \bar{f} + \alpha(f)$. Hence, $\|h - \bar{h}\| = \|f - \bar{f}\|$. By Proposition 3.5 we find that $\|f - \bar{f}\| = \|h - \bar{h}\| = 2\alpha(f)$. Since the hypothesis of the corollary to Theorem 3.1 is satisfied, (3.9) follows. The last equality in (3.9) is obtained by substitution in $\beta(f)$ for $\hat{x}_p$ from (3.2). To show $f(s)-\bar{f}(s)=2\max\{\tilde{f}(s), 0\}$, we note that $\hat{y}_p=\hat{x}_p/2$ and hence $\tilde{f}(s)=\sup\{\hat{x}_p(f): p\in P_s\}/2$. Since $h=f+\alpha(f)$ and $\bar{h}=\bar{f}+\alpha(f)$, substituting for h and $\bar{h}$ in (3.5) we observe that (3.5) holds when h and $\bar{h}$ there are replaced by f and $\bar{f}$. This latter equation may easily be shown to be equivalent to $\bar{f}(s)=\min\{f(s), f(s)-2\tilde{f}(s)\}$ from which the required result follows. The proof is complete.

We remark that the equality $d(f, K)=\|f-\bar{f}\|/2$ is obtained in [22, 23] under different assumptions. To make an observation on translation by constant functions considered in Lemma 3.2 and Propositions 3.5 and 3.6, let $0<\lambda<\infty$ be a function on S. Instead of (3.1) define $x^*_{p,q}$ by

$$x^*_{p,q}(f)=f(s)/\lambda(s)-\sum\{(q(t)/\lambda(t))f(t): t\in A\},$$

where, for all $p=(A, s)$ and $q\in Q_p$, we have $\sigma_{p,q}=1$ and $\sum_A q(t)/(\lambda(t)\,w(t))<\infty$. Then the cone K, defined as before with these new functionals, will be closed under translation by functions of the form $\alpha\lambda$, where α is real. Considering a new weight function $w'=\lambda w$, norm $\|f\|'=\sup\{w'(s)\,|f(s)|: s\in S\}$, space $B'=\{f/\lambda: f\in B\}$, cone $K'=\{k/\lambda: k\in K\}$, one may show that the above problem of finding a best approximation to $f\in B$ from K relative to $\|\cdot\|$ is equivalent to finding a best approximation to $f'=f/\lambda\in B'$ from K' relative to $\|\cdot\|'$. Note that $\|f\|=\|f'\|'$. Clearly, K' is closed under translation by constant functions, since it is defined, as before, by functionals of the form (3.1) with $\sigma_{p,q}=1$. Thus sometimes K may be transformd to K' which is translation-invariant by constant functions. However, the convex cone K of sub-additive functions in Example 4.3 cannot be so transformed to K'.

We now consider the case when K is a convex cone. Let $x^*_{p,q}$ be given by (3.1) and define K by

$$K=\{k\in B: x^*_{p,q}(k)\leqslant 0,\ p\in P,\ q\in Q_p\},\tag{3.10}$$

where Q_p, $p\in P$, is not necessarily finite. It is easy to verify that K is a closed convex cone and $K=\bigcap\{K_{p,q}: p\in P,\ q\in Q_p\}$, where $K_{p,q}$ is as in Proposition 2.2. To place this problem in our earlier format for a nonconvex cone, define a set R of ordered pairs by $R=\{(p, q): p\in P, q\in Q_p\}$. For each $r=(p, q)\in R$, we let $\hat{x}_r=x^*_{r,q}=x^*_{p,q}$ and $Q_r=\{q\}$. We may then write $\hat{x}_r=\min\{x^*_{r,q}: q\in Q_r\}$ and $K=\{f\in B: \hat{x}_r(f)\leqslant 0,\ r\in R\}$, and derive the definitions and results for the convex cone as a special case of the nonconvex cone K. In particular, we obtain from (3.3) and (3.4), respectively, the following:

$$h^0(s) = \min\{h(s), h(s) - \sup\{x^*_{p,q}(h) : p \in P_s, q \in Q_p\}, \qquad s \in S',$$
$$= h(s), \qquad s \in S \setminus S'. \qquad (3.11)$$

$$h^0(s) = \min\left\{h(s), \inf\left\{\sum_A q(t)\, h(t) : p \in P_s, q \in Q_p\right\}, \qquad s \in S'. \qquad (3.12)$$

Also $\alpha(f) = \beta(f) = \sup\{x^*_{p,q}(f)/\|x^*_{p,q}\| : \text{all } p, q\}$. With these observations and recalling that Q_p may be infinite, we have the following theorem.

THEOREM 3.4 (Duality and Lipschitzian selections for convex K). *Assume that $h^0 = h^0_f \in K$ for each $f \in B \setminus K$ with h^0 defined by* (3.11). *Then $h^0 = \bar{h}$ and*

$$d(f, K) = \alpha(f) = \sup\{x^*_{p,q}(f)/\|x^*_{p,q}\| : p \in P, q \in Q_p\}. \qquad (3.13)$$

Furthermore, $f' = \bar{h}$ is the maximal best approximation to f with $\|h - \bar{h}\| \leqslant 2\alpha(f)$. Let

$$\hat{y}_p = \sup\{x^*_{p,q}/\|x^*_{p,q}\| : q \in Q_p\}, \qquad p \in P,$$
$$\bar{f}(s) = \sup\{\hat{y}_p(f) : p \in P_s\}, \qquad s \in S'.$$

Then $\hat{y}_p$ is a positively homogeneous sub-additive functional on B with $\|\hat{y}_p\| = 1$ and the following duality holds:

$$d(f, K) = \sup\{\hat{y}_p(f) : p \in P\} = \sup\{\bar{f}(s) : s \in S'\}, \qquad f \in B \setminus K.$$

The conclusions of the Lipschitzian selection Theorem 3.3 *and Propositions* 3.5 *and* 3.6 *apply under the hypothesis stated there. In particular,* (3.9) *becomes*

$$d(f, K) = \alpha(f) = \|f - \bar{f}\|/2$$
$$= \sup\left\{f(s) - \sum_A q()t)\, f(t) : p = (A, s) \in P, q \in Q_p\right\} / 2, \qquad f \in B \setminus K. \qquad (3.14)$$

and $f(s) - \bar{f}(s) = 2\max\{\bar{f}(s), 0\}$ holds for all $s \in S'$.

Proof. These results may be easily derived from Theorems 3.1–3.3 and Propositions 3.4–3.6 using the argument given above. As in Proposition 2.2, $\hat{y}_p$ is sub-additive. Note that the condition of finiteness of Q_p assumed in these results automatically holds. The proof is complete.

We note that by [18, corollary to Theorem 1], (3.13) implies that the three equivalent conditions (a), (b), and (c) of that theorem hold with $L = \{x^*_{p,q} : \text{all } p, q\}$.

Let S be topological and $w>0$ be a continuous function on S. Let C denote the space of continuous functions f on S with $\|f\|_w<\infty$. We remark on the applicability of the earlier results when $f\in C$. Since $C\subset B$, the duality (3.6) and Theorem 3.2 hold under the hypothesis of Theorem 3.1 when $f\in C$. Now assume additionally that $\bar{f}\in C$ whenever $f\in C$. Since $h=h_f=f+\alpha(f)/w\in C$, by the above assumption, $f'=\bar{h}\in K\cap C$. Consequently, $\bar{h}$ is the maximal best approximation to f from $K\cap C$ in Theorem 3.1. In Theorem 3.3, $T\colon C\to K\cap C$ is Lipschitzian with $\|T(f)-T(g)\|\leqslant c\|f-g\|$ for all $f, g\in C$. Similar remarks may be made on Propositions 3.5 and 3.6 and Theorem 3.4.

4. Approximation Problems

In this section we apply the previous results to approximation problems. For $A\subset S$ we denote by $|A|$ the cardinality of A.

Example 4.1 (Approximation by quasi-convex functions). Let S be a nonempty convex subset of a vector space. A real function f on S is said to be quasi-convex if

$$f(\lambda s+(1-\lambda)t)\leqslant\max\{f(s), f(t)\} \tag{4.1}$$

holds for all $s, t\in S$ and all $0\leqslant\lambda\leqslant 1$ [2, 14]. Without loss of generality we may assume that $s\neq t$ in (4.1). Let K be the set of all quasi-convex functions in B. It is easy to verify that K is a closed cone which is not convex. Let co(A) denote the convex hull of $A\subset S$. By induction or otherwise, it may be easily shown that f is quasi-convex if and only if for every nonempty finite set $A\subset S$ and $s\in\mathrm{co}(A)$ we have

$$f(s)\leqslant\max\{f(t)\colon t\in A\}. \tag{4.2}$$

Clearly, we may assume that $s\notin A$ in (4.2). In that case $|A|\geqslant 2$.

To place this problem in our earlier format of Section 3, let $\mathscr{A}$ be the set of all finite subsets of A of S with $|A|\geqslant 2$ and $\tau(A)=\mathrm{co}(A)\backslash A$. It is easy to verify that $S'=S\backslash E$, where E is the set of extreme points of S [14]. For each $u\in A$, define functions q_u on A as follows: $q_u(t)=1$ if $t=u$, $=0$ if $t\in A$ and $t\neq u$. Then, if $p=(A, s)$ where $s\in\mathrm{co}(A)\backslash A$, we let $Q_p=\{q_u\colon u\in A\}$ and define $x^*_{p,q}$ and $\hat{x}_p$ by (3.1) and (3.2), respectively. We then have

$$x^*_{p,q}(f)=f(s)-\sum_{t\in A} q_u(t)\,f(t)=f(s)-f(u), \qquad \text{if} \quad q=q_u,$$

$$\hat{x}_p(f)=f(s)-\max\{f(u)\colon u\in A\}.$$

Clearly, $K=\{k\in B: \hat{x}_p(k)\leqslant 0,\ \ p\in P\}$. By Proposition 3.2 we obtain $\|x^*_{p,q}\| = 1/w(s)+1/w(u)$ if $q=q_u$, and $\|\hat{x}_p\| = 1/w(s)+\max\{1/w(u): u\in A\} < \infty$. Since $\sigma_{p,q}=1$ for all p, q we have $\sigma=1$.

To establish the next lemma, note that P_s is the set of all $p=(A, s)$ such that $s\in \operatorname{co}(A)\backslash A$. For any $h\in B$, we define h^0 by (3.3) or (3.4), i.e.,

$$\begin{aligned} h^0(s) &= \min\{h(s), \inf\{\max\{h(u): u\in A\}: p\in P_s\}\}, && s\in S', \\ &= h(s), && s\in S\backslash S'. \end{aligned} \tag{4.3}$$

Clearly, $h^0\leqslant h<\infty$. Note that, in general, $h^0>-\infty$ is not true even if $h\in B$. Since we are only interested in real functions, we impose the condition $h^0>-\infty$ in the next lemma. However, it may be easily shown that h^0 satisfies (4.1) even if $h^0(s)=-\infty$ for some $s\in S$. Similar comments apply to corresponding lemmas for other examples in this article.

LEMMA 4.1. *If h^0 defined by (4.3) satisfies $h^0>-\infty$, then it is quasi-convex.*

Proof. For convenience denote h^0 by k. Let $s, t\in S$ with $s\neq t$, $x=\lambda s+(1-\lambda)t$ where $0<\lambda<1$, and $\varepsilon>0$. Note that x cannot be an extreme point of S and hence $x\in S'$. We show that k satisfies $\max\{k(s), k(t)\}\geqslant k(x)$, which is (4.1).

Suppose first that $s, t\in S'$. Then there exist $(A, s)\in P_s$ and $(D, t)\in P_t$ such that

$$k(s)\geqslant \min\{h(s), \max\{h(u): u\in A\}-\varepsilon\}, \tag{4.4}$$

$$k(t)\geqslant \min\{h(t), \max\{h(u): u\in D\}-\varepsilon\}. \tag{4.5}$$

Suppose that the minimum in each of (4.4) and (4.5) is attained at the second term on its right-hand side. Then with $F=A\cup D$ we have

$$\max\{k(s), k(t)\}\geqslant \max\{h(u): u\in F\}-\varepsilon = M, \tag{4.6}$$

say. Clearly, $x\in\operatorname{co}(F)$. We now have two cases. If $x\in F$, then $M\geqslant h(x)-\varepsilon\geqslant k(x)-\varepsilon$. If, on the other hand, $x\in\operatorname{co}(F)\backslash F$, then $(F, x)\in P_x$ and $M\geqslant k(x)-\varepsilon$ by the definition of k. Now suppose that the minimum in, say, (4.4) is attained at the second term on its right-hand side, and in (4.5) at $h(t)$. Then $k(t)\geqslant h(t)$ and hence $k(t)=h(t)$. In this case again (4.6) holds with $F=A\cup\{t\}$. Now considering the two cases $x\in F$ and $x\notin F$ as above, we conclude that $M\geqslant k(x)-\varepsilon$. If in (4.4) and (4.5), the minimum is attained at $h(s)$ and $h(t)$, respectively, then $k(s)=h(s)$ and $k(t)=h(t)$ as above. If $F=\{s, t\}$, then clearly, $x\in\operatorname{co}(F)\backslash F$ and, hence, $(F, x)\in P_x$. We then have $\max\{k(s), k(t)\}=\max\{h(u): u\in F\}\geqslant k(x)$. We have shown that (4.1) holds if $s, t\in S'$. The remaining cases for which $s\in S'$, $t\in S\backslash S'=E$, and $s, t\in E$ may be analyzed as above. The proof is complete.

A special case of the above lemma occurs when $h = h_f = f + \alpha(f)/w$ for $f \in B \backslash K$. Since by Proposition 3.4, $h^0 = h_f^0 \in B$ for all $f \in B \backslash K$, we conclude that $h^0 \in K$ and $h^0 = \bar{h}$. Consequently, Theorems 3.1 3.3 apply with $\sigma = 1$ and $c = 2$. In particular, using the values of $\|x^*_{p,q}\|$, we obtain from (3.6) the duality

$$d(f, K) = \alpha(f) = \sup\left\{\min\left\{\frac{w(s)\,w(t)}{w(s)+w(t)}\,(f(s) - f(t)) \colon t \in A\right\}\right\}, \qquad f \in B \backslash K,$$

where the supremum is taken over all (A, s) with $A \in \mathscr{A}$ and $s \in \mathrm{co}(A) \backslash A$. If $w = 1$, then Proposition 3.6 applies and, by (3.9), we have

$$\begin{aligned} d(f, K) = \alpha(f) = \beta(f) &= \sup\{f(s) - \max\{f(t) \colon t \in A\}\}/2 \\ &= \| f - \bar{f} \|/2, \qquad f \in B \backslash K, \end{aligned}$$

where the supremum is taken as above. Other forms of duality may be obtained from Theorem 3.2.

An explicit expression for the greatest K-minorant $\bar{h}$ of a bounded function h on S (i.e., with $w = 1$) is obtained in [21]. This expression is valid when $h \in B$, as may be easily shown. See also [23] in this connection. Note that (4.3) gives another expression for $\bar{h}$. Rewriting (4.3) we obtain $\bar{h}(s) = \inf\{\max\{h(t)\} \colon t \in D\}\}$, $s \in S$, where the infimum is taken over all finite subsets D of S such that $|D| \geqslant 1$ and $s \in \mathrm{co(D)}$. We now consider a convex $S \subset R^n$ and obtain stronger results using the well-known Caratheodory's Theorem.

PROPOSITION 4.1. *If* $S \subset R^n$, *then it suffices to consider* $A \subset S$ *with* $2 \leqslant |A| \leqslant n+1$ *in the above analysis, and, in particular, in* (4.3).

Proof. Clearly, we may take $|A| \leqslant n+1$ in (4.2) and, hence, in the definition of $x^*_{p,q}$ and $\hat{x}_p$. Now consider (4.3) and let $s \in S'$. For convenience let $\mu = \inf\{\max\{h(u) \colon u \in A\} \colon p \in P_s\}$. Let $p = (A, s) \in P_s$. Since $s \in \mathrm{co}(A)$, by Caratheodory's Theorem [15], there exists $D \subset A$ with $|D| \leqslant n+1$ such that $s \in \mathrm{co}(D)$. Then $(D, s) \in P_s$ and $\mu \leqslant \max\{h(u) \colon u \in D\} \leqslant \max\{h(u) \colon u \in A\}$. It follows that we may consider only those $A \in \mathscr{A}$ with $|A| \leqslant n+1$ in (4.3). The proof is complete.

Recall that the space C is defined at the end of Section 3. If $S \subset R^n$ is compact and h is in C, then is its greatest K-minorant $\bar{h}$ is in C. A proof of this is as in [23]. Consequently, the remarks made at the end of Section 3 are applicable.

EXAMPLE 4.2 (Approximation by convex functions). Let S be a nonempty convex subset of a vector space. A real function f of S is said to be convex if

$$f(\lambda s + (1-\lambda)t) \leqslant \lambda f(s) + (1-\lambda)\, f(t)$$

holds for all $s, t \in S$ and all $0 < \lambda < 1$. Without loss of generality, let $s \neq t$ in the above definition. Let K be the set of all convex functions in B. It is easy to verify that K is a closed convex cone. By induction or otherwise it may be easily shown that k is convex if and only if for every nonempty finite set $A \subset S$ and every positive real function q on A with $\sum \{q(t): t \in A\} = 1$ and $s = \sum \{q(t)t: t \in A\}$ we have

$$f(s) \leqslant \sum \{q(t) f(t): t \in A\}. \tag{4.7}$$

See, e.g., [14, 15].

LEMMA 4.2. *It suffices to assume that $s \notin A$ in definition* (4.7).

Proof. We show that (4.7) with $s \notin A$ implies (4.7) with $s \in A$. Assume that $s \in A$, $s = \sum_A q(t)t$ for some $q(t) > 0$ for all t in A with $\sum_A q(t) = 1$. Let $D = A \setminus \{s\}$ and $\mu = q(s)$. If $q'(t) = q(t)/(1-\mu)$, $t \in D$, then $q'(t) > 0$ and $\sum_D q'(t) = 1$. Since $s = \sum_A q(t)t$, a minor rearrangement of terms shows that $s = \sum_D q'(t)t$. Again, since $s \notin D$, we have $f(s) \leqslant \sum_D q'(t) f(t)$. Now it is easy to verify that

$$\sum_A q(t) f(t) = (1-\mu) \sum_D q'(t) f(t) + \mu f(s) \geqslant (1-\mu) f(s) + \mu f(s) = f(s).$$

The proof is complete.

Let $\mathscr{A}$ be the set of all finite subsets A of S with $|A| \geqslant 2$. As justified by Lemma 4.2, we assume that $s \notin A$, and hence $|A| \geqslant 2$. Let $A \in \mathscr{A}$. We say that $s \in S$ is a positive convex combination of all elements in A if $s = \sum_A q(t)t$ where $q(t) > 0$ for all $t \in A$ and $\sum_A q(t) = 1$. Let $\operatorname{po}(A)$ denote the set of all positive convex combinations of all elements in A. Then, clearly $\operatorname{po}(A)$ is convex and $\operatorname{po}(A) \subset \operatorname{co}(A)$. Let $\tau(A) = \operatorname{po}(A) \setminus A$. We assert that $S' = S \setminus E$, where E is as in Example 4.1. This follows because $u \in S \setminus E$ if and only if there exist $s, t \in S$ and $0 < \lambda < 1$ with $u = \lambda s + (1-\lambda)t$, and then $u \in \tau(A)$ where $A = \{s, t\} \in \mathscr{A}$. For each $p = (A, s)$ with $s \in \operatorname{po}(A) \setminus A$, define Q_p to be the set of all functions q on A with $q > 0$, $\sum_A q(t) = 1$, and $s = \sum_A q(t)t$. Recall that Q_p may be allowed to be infinite for a convex cone K. Define $x^*_{p,q}$ by (3.1). Then, $K = \{f \in B: x^*_{p,q}(f) \leqslant 0$ for all $p, q\}$. By Proposition 3.3, we obtain, $\|x^*_{p,q}\| = 1/w(s) + |q|$. Again $\sigma_{p,q} = 1$ for all p, q and, hence, $\sigma = 1$. Note that P_s is the set of all $p = (A, s)$ with $s \in \operatorname{po}(A) \setminus A$. Recall the remarks preceding Lemma 4.1.

LEMMA 4.3. *If h^0 defined by* (3.11) *or* (3.12) *satisfies $h^0 > -\infty$, then it is convex.*

Proof. We let $k = h^0$ for convenience. Let $s, t \in S$ with $s \neq t$, $x = \lambda s + (1-\lambda) t$ where $0 < \lambda < 1$, and $\varepsilon > 0$. As in Example 4.1, $x \in S'$. We show that $\lambda k(s) + (1-\lambda) k(t) \geqslant k(x)$. The various cases to be considered in this proof are as in Lemma 4.1. We omit similar details while emphasizing the differences.

Suppose first that $s, t \in S'$. Then, by (3.12), there exist $p = (A, s) \in P_s$, $p' = (D, t) \in P_t$, $q \in Q_p$, and $q' \in Q_\rho$ $(\rho = p')$ such that $s = \sum_A q(u) u$, $t = \sum_D q'(u) u$, and

$$k(s) \geqslant \min \left\{ h(s), \sum_A q(u)\, h(u) - \varepsilon \right\},$$

$$k(t) \geqslant \min \left\{ h(t), \sum_D q'(u)\, h(u) - \varepsilon \right\}.$$

Suppose first that the minimum in each of the above two inequalities is attained at the second term on its right-hand side. Let $F = A \cup D$ and define $r(u)$ for $u \in F$ by

$$\begin{aligned} r(u) &= \lambda q(u) + (1-\lambda)\, q'(u), && \text{if } u \in A \cap D, \\ &= \lambda q(u), && \text{if } u \in A \setminus D, \\ &= (1-\lambda) q'(u), && \text{if } u \in D \setminus A. \end{aligned}$$

Then $r > 0$, $\sum_F r(u) = 1$, and $x = \Sigma_F\, r(u) u$. Clearly, $x \in \operatorname{po}(F)$ and we have

$$\lambda k(s) + (1-\lambda)\, k(t) \geqslant \sum_F r(u)\, h(u) - \varepsilon = M,$$

say. If $x \in \operatorname{po}(F) \setminus F$, then $(F, x) \in P_x$ and $M \geqslant k(x) - \varepsilon$ by the definition of k. Now suppose that $x \in F$. Define $G = F \setminus \{x\}$, $\mu = r(x)$, and $r'(u) = r(u)/(1-\mu)$ for $u \in G$. Then $r' > 0$, $\sum_G r'(u) = 1$, and $x = \sum_G r'(u) u$. Since $x \in \operatorname{po}(G) \setminus G$, we find that $(G, x) \in P_x$ and $\sum_G r'(u)\, h(u) \geqslant k(x)$. Now a minor computation as in Lemma 4.2 combined with $h(x) \geqslant k(x)$ gives us

$$\sum_F r(u)\, h(u) = (1-\mu) \sum_G r'(u)\, h(u) + \mu h(x) \geqslant (1-\mu)\, k(x) + \mu k(x) = k(x).$$

Hence $M \geqslant k(x) - \varepsilon$. The remaining cases are as in Lemma 4.1 and their proof is as given above. The proof is complete.

A special case of the above lemma occurs when $h = h_f = f + \alpha(f)/w$ for $f \in B \setminus K$. By a version of Proposition 3.4 as applied to convex K, we conclude that $h^0 = h_f^0 \in K$ for all $f \in B \setminus K$, and hence $h^0 = \bar{h}$. Consequently

Theorem 3.4 applies with $\sigma = 1$ and $c = 2$. In particular, using the value of $\|x^*_{p,q}\|$ we obtain from (3.13) the duality

$$d(f, K) = \alpha(f)$$
$$= \sup\left\{\frac{f(s) - \sum_A q(t)\, f(t)}{1/w(s) + |q|} : p = (A, s) \in P,\ q \in Q_p\right\}, \qquad f \in B \setminus K. \qquad (4.8)$$

If $w = 1$, we obtain from (3.14) the following for $f \in B \setminus K$:

$$d(f, K) = \alpha(f)$$
$$= \sup\left\{f(s) - \sum_A q(t)\, f(t) : p = (A, s) \in P,\ q \in Q_p\right\} / 2 = \|f - \bar{f}\|/2. \qquad (4.9)$$

The equality, $d(f, K) = \|f - \bar{f}\|/2$, appearing in (4.9) is established in [22, 23] by different methods. This equality and [15, Theorem 5.6] with the substitution $f = \bar{f}$ and $f_i = f$ give an alternative proof of (4.9).

An explicit expression for the greatest K-minorant $\bar{h}$ of $h \in B$ may be obtained as below. The epigraph $E(h)$ of h is defined by

$$E(h) = \{(s, \mu) \in S \times R : \mu \geqslant h(s)\}. \qquad (4.10)$$

Then

$$\bar{h}(s) = \inf\{\mu : (s, \mu) \in \mathrm{co}(E(h))\}.$$

For a proof see [15, p. 36]. Another expression for $\bar{h}$ may be based on (3.12) or [15, Theorem 5.6] as observed above.

We now obtain stronger results for a convex $S \subset R^n$.

PROPOSITION 4.2. *If* $S \subset R^n$, *then it suffices to consider* A *with* $2 \leqslant |A| \leqslant n+1$ *in the above analysis and, in particular, in the definition* (3.11) *or* (3.12) *of* h^0.

Proof. Clearly, we may take $|A| \leqslant n+1$ in (4.7) and hence in the definition of $x^*_{p,q}$. Now consider (3.12). Let $s \in S'$, $p \in P_s$, and $q \in Q_p$. For convenience, let $\theta = \sum_A q(t)\, h(t)$ and $\mu = \inf\{\sum_A q(t)\, h(t) : p \in P_s, q \in Q_p\}$. We then have $\mu \leqslant \theta$ and $(s, \theta) \in \mathrm{po}(A') \subset \mathrm{co}(A')$, where $(s, \theta) \in R^{n-1}$ and $A' = \{(t, h(t)) : t \in A\} \subset R^{n+1}$. By Caratheodory's Theorem [15, Corollary 17.1.1], there exists $E \subset A'$ with $|E| \leqslant n+2$ such that points in E are affinely independent and $(s, \theta) \in \mathrm{co}(E)$. Then $\mathrm{co}(E)$ is a simplex in R^{n+1}. We now argue as in the proof of [15, Corollary 17.1.3]. Since (s, θ) is in the simplex, there is a minimal $\theta' \leqslant \theta$ such that (s, θ') is in the same simplex. Then, as in that proof, we may find $D \subset A$ with $|D| \leqslant n+1$ such that if $D' = \{(t, h(t)) : t \in D\}$ then $(s, \theta') \in \mathrm{co}(D')$. Thus, $\mathrm{co}(D')$ is a subsimplex of co(E). Then $s=\sum_D \lambda(t)t$, $\theta'=\sum_D \lambda(t)h(t)$ for some $\lambda(t)>0$ and $\sum_D \lambda(t)=1$. (If $\lambda(t)=0$ for some t, then we may delete that pint from D.) Since $s\notin A$, $s\notin D$. Thus $(D, s)\in P_s$ and $\mu\leqslant\theta'\leqslant\theta$. It follows that we may include only those A in (3.12) such that $|A|\leqslant n+1$. The proof is complete.

If $S\subset R^n$ is a polytope and h is in C, then its greatest K-minorant $\bar{h}$ is in C. (A polytope is a convex hull of a finitely many points and hence, is compact.) This assertion may be proved as in [23]. Consequently, the remarks made at the end of Section 3 are applicable here. The problem of this section on C when $n=1$ and $w=1$ is considered in [25].

We make a remark on Example 4.1. It may be easily shown that f is quasi-convex if and only if (4.2) holds for every finite set $A\subset S$ and $s\in \text{po}(A)\backslash A$. Hence, we may define $\tau(A)=\text{po}(A)\backslash A$ and replace co(A) by po(A) everywhere in that example.

EXAMPLE 4.3 (Approximation by sub-additive functions). Let $S=(0, b)$, where $0<b\leqslant\infty$, be a real interval. A real function f on S is said to be sub-additive if

$$f(s+t)\leqslant f(s)+f(t)$$

holds for all $s, t\in S$ with $s+t\in S$ [9, 16]. Let K be the set of all sub-additive functions in B. It is easy to verify that K is a closed convex cone. A function q on a subset of S is called positive integral if its range is positive integers. By induction or otherwise it may be easily shown that f is sub-additive if and only if for every nonempty finite set $A\subset S$ and positive integral function q on A with $s=\sum_A q(t)t\in S$ we have

$$f(s)\leqslant\sum\{q(t)f(t): t\in A\}.$$

We may assume that $s\notin A$ in the above definition. Because if $s\in A$, then $s=\sum_A q(t)t$ implies that $A=\{s\}$ and $q(s)=1$. Since 0 is sub-additive but -1 is not, we conclude that K is not closed under translation by constant functions. The remarks following Proposition 3.6 are applicable here.

We say that $s\in R$ is a positive integral combination of all elements in A if $s=\sum_A q(t)t$ for some positive integral function q on A. Let pi(A), called the positive integral hull of A, denote the set of all positive integral combinations s of all elements in A such that $s\in S$. This set corresponds to po(A) in Example 4.2. Let $\mathscr{A}$ be the set of all nonempty finite subsets A of S such that $\text{pi}(A)\backslash A\neq\varnothing$. Note that if A in $\mathscr{A}$ is a singleton and equals $\{t\}$ then $2t\in S$, otherwise $\text{pi}(A)\backslash A=\varnothing$. If $|A|\geqslant 2$, then the sum of elements in A is in S. Let $\tau(A)=\text{pi}(A)\backslash A$. We assert that $S'=S$. To see this for s in S, let $A=\{t\}$, where $t=s/2$. Then $s\in\tau(A)$ and $S'=S$. For each $p=(A, s)\in P$ with $s\in\text{pi}(A)\backslash A$, define Q_p to be the set of all positive integral functions q on A such that $s=\sum_A q(t)t$. By the above stated condition on A, we have $Q_p \neq \varnothing$; since, if $|A| \geqslant 2$, then Q_p contains the unity function $q=1$ on A, otherwise, if $|A|=1$, then Q_p contains the function $q=2$ on A. Define $x^*_{p,q}$ by (3.1). Then $K=\{f \in B: x^*_{p,q}(f) \leqslant 0, \text{ all } p, q\}$. Clearly, $\sigma_{p,q} \geqslant 2$ for all p, q and, hence, $\sigma \geqslant 2$. It is easy to show that $\sigma=\infty$.

LEMMA 4.4. *If h^0 defined by* (3.11) *or* (3.12) *satisfies $h^0 > -\infty$, then it is sub-additive.*

Proof. The proof is similar to that of Lemma 4.3, however, much simpler. For example, if s, t and $x=s+t$ are in S, and A and D are sets corresponding to s and t as in Lemma 4.3, then it may be easily seen that $x \notin F=A \cup D$. Hence $x \in \text{pi}(F)\backslash F$ and the case $x \in F$ does not occur. We omit further details. The outline of the proof is complete.

Arguing as in Example 4.2, we conclude that the duality in Theorem 3.4 applies. In particular, (4.8) and (4.9) hold with appropriate interpretations of P and Q_p. Since $c=\sigma=\infty$, the Lipschitzian selection of Theorem 3.4 does not apply.

We now obtain an explicit expression for the greatest K-minorant $\bar{h}$ of $h \in B$. We first introduce some difinitions. A subset H of $S \times R$ is called integral if whenever (s, λ), $(t, \mu) \in H$ and $s+t \in S$, then $(s+t, \lambda+\mu) \in H$. The next lemma justifies this terminology. Let $\mathscr{H}$ denote all the integral subsets H of $S \times R$. Clearly, $S \times R \in \mathscr{H}$. It is easy to see that $\mathscr{H}$ is closed under arbitrary intersections. Hence, given $G \subset S \times R$ there exists a smallest set in $\mathscr{H}$ containing G, which is the intersection of all the sets in $\mathscr{H}$ containing G. It is called the integral hull of G and is denoted by in (G). The results of the next lemma are similar to those in convexity theory [14]; an integral set corresponds to a convex set and $\text{in}(G)$ to the convex hull $\text{co}(G)$.

LEMMA 4.5. (a) *A* subset *H of $S \times R$ is integral if and only if the following holds. For every finite subset $\{(s_i, \lambda_i): 1 \leqslant i \leqslant n\} \subset H$ and positive integers m_i, $1 \leqslant i \leqslant n$, if $\sum m_i s_i \in S$, then $(\sum m_i s_i, \sum m_i \lambda_i) \in H$.*

(b) *Suppose $G \subset S \times R$. Then $(s, \lambda) \in \text{in}(G)$ if and only if there exists a finite subset $\{(s_i, \lambda_i): 1 \leqslant i \leqslant n\} \subset G$ and positive integers m_i such that $s=\sum m_i s_i$ and $\lambda=\sum m_i \lambda_i$.*

Proof. The proof follows directly from the definitions as in convexity theory. See, e.g., Theorems B and D of [14, p. 75]. Part (i) may be proved by induction on n and (ii) by using (i). The outline of the proof is complete.

Recall that the epigraph of a function is defined by (4.10).

LEMMA 4.6. *A function f in B is sub-additive if and only if its epigraph $E(f)$ is integral.*

Proof. Let f be sub-additive and $(s_i, \lambda_i) \in E(f)$, $1 \leqslant i \leqslant n$, with $\sum m_i \lambda_i \in S$ for some positive integers m_i. Then $\lambda_i \geqslant f(s_i)$. Now $f(\sum m_i s_i) \leqslant \sum m_i f(s_i) \leqslant m_i \lambda_i$. Hence, $(\sum m_i s_i, \sum m_i \lambda_i) \in E(f)$ and $E(f) \in \mathscr{H}$ by Lemma 4.5. Conversely, let $E(f) \in \mathscr{H}$ and $s, t \in S$ with $s + t \in S$. Then $(s, f(s))$, $(t, f(t)) \in E(f)$. Hence $(s + t, f(s) + f(t)) \in E(f)$ which implies that $f(s) + f(t) \geqslant f(s + t)$. The proof is complete.

PROPOSITION 4.3. *If $\bar{h}$ is the greatest K-minorant of $h \in B$, then*

$$\bar{h}(s) = \inf\{\lambda : (s, \lambda) \in \operatorname{in}(E(h))\}, \qquad s \in S.$$

Proof. Let $k(s)$ denote the right-hand side of (4.11). We show that $\bar{h} = k$. By definition, $\bar{h} \in K \subset B$. Since $\bar{h} \leqslant h$ we have $E(\bar{h}) \supset E(h)$. By Lemma 4.6, $E(\bar{h}) \in \mathscr{H}$. Hence $E(\bar{h}) \supset \operatorname{in}(E(h)) \supset E(h)$. It follows that $\bar{h} \leqslant k \leqslant h$ and, hence, $k \in B$. Since $\operatorname{in}(E(h)) \in \mathscr{H}$, by Lemma 4.6, k is sub-additive, and hence $k \in K$. By the definition of $\bar{h}$ we have $k \leqslant \bar{h}$ and thus $k = \bar{h}$. The proof is complete.

Another expression for $\bar{h}$ may be obtained from (3.12).

5. EXTENSIONS AND APPLICATIONS

In this section, we extend previous results to more complex cones and indicate applications to approximation problems.

Let $K = \{f \in B : \hat{x}_p(f) \leqslant 0,\ p \in P\}$ as in Section 3 with finite Q_p for each $p \in P$, and $K' = \{f \in B : x^*_{p,q}(f) \leqslant 0,\ p \in P',\ q \in Q'_p\}$, where P' and Q'_p, $p \in P'$, are index sets and Q'_p may be infinite. We are interested in finding a best approximation from the cone $K \cap K'$. The results for this case may be easily obtained by arguing as for the convex cone in Section 3. Specifically, h^0 equals the minimum of the right-hand sides of (3.3) and (3.11) with the change that P_s and Q_p in (3.11) are respectively replaced by P'_s and Q'_p. Similarly, $\alpha(f)$ equals the maximum of the right-hand sides of (3.6) and (3.13), again, with P and Q_p in (3.13) replaced by P' and Q'_p, respectively. Obvious simplifications may be made in both expressions. Modifications for other results are similar. Now suppose that $E \subset S$, $K'' = \{f \in B : f(s) \leqslant 0,\ s \in E\}$ and approximation is from $K \cap K' \cap K''$. We may then let $x^*_s(f) = f(s)$, $s \in E$. This functional may be considered as a special case of $x^*_{p,q}$ where $p = (A, s)$ and q is the zero function on A. As in Proposition 3.2 we have $\|x^*_s\| = 1/w(s)$. This functional may be handled by letting $h^0(s) \leqslant 0$ for all $s \in E$.

EXAMPLE 5.1. To illustrate an application, we let S be any set with partial order $\leqslant$. A partial order on S is a reflexive and transitive relation on S which is not necessarily anti-symmetric [7]. If $s, t \in S$ and $s \leqslant t$ but

$s \neq t$ then we write $s < t$. Let E and F be subsets of S and K be the set of all f in B satisfying

$$f(s) \leqslant f(t), \qquad s, t \in S \text{ and } s < t, \tag{5.1}$$

$$f(s) \leqslant 0, \qquad s \in E, \tag{5.2}$$

$$f(s) \geqslant 0, \qquad s \in F. \tag{5.3}$$

The problem is to find a best approximation to f from K. If $G = \{s \in S : s \leqslant t \text{ for some } t \in E\}$ and $H = \{t \in S : s \leqslant t \text{ for some } s \in F\}$, then (5.2) and (5.3) are respectively equivalent to

$$f(s) \leqslant 0, \qquad s \in G, \tag{5.4}$$

$$f(s) \geqslant 0, \qquad s \in H. \tag{5.5}$$

It is easy to show that K is a closed convex cone. We consider the special cases of K later. This problem has been considered by different methods in [19]. We show that it falls in the framework of this article. We also identify Lipschitzian selections which are not considered in [19]. For the purpose of analysis, we define K_1 (resp. K_2) to be the set of all f in B satisfying (5.1) and (5.4) (resp. (5.5)). Both K_1 and K_2 are closed convex cones and $K = K_1 \cap K_2$.

To apply the results of Section 3 to K_1, let $\mathscr{A}$ consist of all the singleton subsets $A = \{t\}$ of S such that there exists $s \in S$ with $s < t$. Then let $\tau(\{t\}) = \{s \in S : s < t\}$. Define $P = \{(\{t\}, s) : \{t\} \in \mathscr{A},\ s \in \tau(\{t\})\}$. If $p = (\{t\}, s) \in P$, then let $Q_p = \{q\}$, where $q(t) = 1$. This gives $x^*_{p,q}(f) = f(s) - f(t)$, which corresponds to (5.1). As observed before, (5.4) gives the functional $x^*_s(f) = f(s) \leqslant 0$, $s \in G$. By Proposition 3.2, these two functionals have the norms $1/w(s) + 1/w(t)$ and $1/w(s)$ respectively. Similarly, K_2 may be analyzed by symmetric methods by considering the inequalities $x^*_{p,q}(f) \geqslant 0$ instead of $x^*_{p,q}(f) \leqslant 0$. In the following proposition, h^0 is obtained by applying (3.11) or (3.12) to K_1 and letting $h^0(s) \leqslant 0$ on G. Similarly, k^0 is obtained from a symmetric version of (3.11) as applied to K_2. Note that $\sigma = 1$. We denote the characteristic function of a set D by χ_D, and let $0 \cdot \infty = 0 \cdot (-\infty) = 0$.

LEMMA 5.1. *Let $h, k \in B$ and define*

$$h^0(s) = \min\{\inf\{h(t) : t \in S, s \leqslant t\}, \infty(1 - \chi_G(s))\}, \qquad s \in S, \tag{5.6}$$

$$k^0(s) = \max\{\sup\{h(t) : t \in S, t \leqslant s\}, -\infty(1 - \chi_H(s))\}, \qquad s \in S. \tag{5.7}$$

If $h^0 > -\infty$, then h^0 satisfies (5.1) *and* (5.4). *Similarly, if $k^0 < \infty$, then k^0 satisfies* (5.1) *and* (5.5).

Proof. The proof is straightforward and hence omitted.

Define,

$$\theta_1(f) = \sup\left\{\frac{w(s)\,w(t)}{w(s)+w(t)}\,(f(s)-f(t)): s,\, t \in S,\, s < t\right\},$$

$$\theta_2(f) = \sup\{w(s)\,f(s): s \in G\},$$

$$\theta_3(f) = \sup\{-w(s)\,f(s): s \in H\}.$$

The above three numbers are $\sup\{x^*(f)/\|x^*\|\}$ corresponding to (5.1), (5.4), and (5.5) respectively. Let $\alpha(f) = \max\{\theta_i(f): 1 \leqslant i \leqslant 3\}$, $\alpha_1(f) = \max\{\theta_1(f), \theta_2(f)\}$, and $\alpha_2(f) = \max\{\theta_1(f), \theta_3(f)\}$. Note that $\alpha_i(f)$ corresponds to K_i as in (3.13), $1 \leqslant i \leqslant 2$.

PROPOSITION 5.1. *Let $f \in B \backslash K$, $h = f + \alpha(f)/w$, and $k = f - \alpha(f)/w$. Let also h^0 and k^0 be defined by* (5.6) *and* (5.7) *respectively for these h and k. Then $\bar{h} = h^0$ and $\underline{k} = k^0$, where $\bar{h}$ is the greatest K-minorant of h and $\underline{k}$ is the smallest K-majorant of k. The duality, $d(f, K) = \alpha(f)$, holds, and $\bar{h}$ and $\underline{k}$ are, respectively, the maximal and minimal best approximation to f with $\underline{k} \leqslant \bar{h}$. For $f \in K$, let $\bar{h} = \underline{k} = f$. Then $g \in K$ is a best approximation to $f \in B$ if and only if $\underline{k} \leqslant g \leqslant \bar{h}$. If $0 \leqslant \lambda \leqslant 1$, then $f' = \lambda\bar{h} + (1-\lambda)\underline{k}$ is a best approximation to $f \in B$ and the operator $T_\lambda: B \to K$ defined by $T_\lambda(f) = f'$ is Lipschitzian with $c = 2$ for all $0 \leqslant \lambda \leqslant 1$.*

Proof. Since $\alpha(f) \geqslant \alpha_1(f)$, by Lemma 5.1 and the corollary to Proposition 3.4 as applied to the convex cone K_1 we conclude that $\bar{h} = h^0$, where $\bar{h}$ is the greatest K_1-minorant of h and $\|h - \bar{h}\| \leqslant 2\alpha(f)$. Similarly, since $\alpha(f) \geqslant \alpha_2(f)$, by Lemma 5.1 and a symmetric version of the corollary applied to K_2, we conclude that $\underline{k} = k^0$, where $\underline{k}$ is the smallest K_2-majorant of f and $\|k - k\| \leqslant 2\alpha(f)$. It is easy to establish by some simple computations that $\underline{k} \leqslant \bar{h}$. Hence $\underline{k} \leqslant 0$ on G and $\bar{h} \geqslant 0$ on H. It follows that $\underline{k}, \bar{h} \in K$. Since $K \subset K_1$, we find that $\bar{h}$ is the greatest K-minorant of h. Similarly, $\underline{k}$ is the smallest K-majorant of k. Clearly, $d(f, K_1) \geqslant \alpha_1(f)$ and $d(f, K_2) \geqslant \alpha_2(f)$. (Using Theorem 3.4 we may show that equalities hold here.) Hence, by Proposition 2.3, $d(f, K) \geqslant \max\{d(f, K_i): 1 \leqslant i \leqslant 2\} \geqslant \alpha(f)$. Since $f = h - \alpha(f)/w$, by arguing as in Theorem 3.1, we have $\|f - \bar{h}\| \leqslant \alpha(f)$. Hence, $\bar{h}$ is a best approximation to f. It is the maximal best approximation because $\bar{h}$ is the greatestt K-minorant of h. Similarly, $\underline{k}$ is the minimal best approximation to f. The remaining statements follow easily. Since $\sigma = 1$ we have $c = 2$. The proof is complete.

We now consider special cases of Example 5.1. If $S = \times\{[a_i, b_i]: 1 \leqslant i \leqslant n\}$ is a rectangle in R^n and the partial order on S is the usual order on R^n, then the problem is the monotone approximation problem with constraints (5.2) and (5.3). Let $S=[0, b]$ be a real interval. A function f on S is called star-shaped if $f(\alpha s) \leqslant \alpha f(s)$ for all $0 \leqslant s \leqslant b$ and $0 \leqslant \alpha \leqslant 1$ [1, 11]. It can be easily shown that f is star-shaped if and only if $f(0) \leqslant 0$ and $f(s)/s \leqslant f(t)/t$ for all $0 < s \leqslant t \leqslant b$. As shown in [19], this problem is a special case of Example 5.1.

We now obtain the duality result for the problem of approximation by quasi-convex functions (Example 4.1) on a real interval I. It has a richer structure than that of Example 4.1 on R^n for $n \geqslant 2$. See [21] for details. This problem with $w=1$ on a compact real interval was considered in [20], however, the arguments for any interval are similar. We use the notation of [20] applied to an arbitrary I. We consider I with the partial order P_x^- (resp. P_x^+) for each $x \in I$ and apply Proposition 5.1 to convex cones K_x^- (resp. K_x^+) to obtain the following duality results. Let $m(s, t) = w(s)\, w(t)/(w(s)+w(t))$. Then,

$$\begin{aligned} d(f, K_x^-) = \theta_x^- &= \max\{\sup\{m(s, t)(f(t)-f(s)) : s, t \in I, s \leqslant t \leqslant x\}, \\ &\qquad \sup\{m(s, t)(f(s)-f(t)) : s, t \in I, x < s \leqslant t\}\} \\ d(f, K_x^+) = \theta_x^+ &= \max\{\sup\{m(s, t)(f(t)-f(s)) : s, t \in I, s \leqslant t < x\}, \\ &\qquad \sup\{m(s, t)(f(s)-f(t)) : s, t \in I, x \leqslant s \leqslant t\}\}. \end{aligned}$$

Since $K = \bigcup \{K_x^- \cup K_x^+ : x \in I\}$, Proposition 2.3 gives us the duality for K as follows.

$$d(f, K) = \inf\{\min\{\theta_x^-, \theta_x^+\} : x \in I\}. \tag{5.8}$$

Now, the results for Example 4.1 applied to this special case show that $\bar{h}$, the greatest K-minorant, is the maximal best approximation. When $w=1$, (5.8) is essentially included in [20, Theorem 4.2]. If $f \in C$, then $\theta_x^- = \theta_x^+$, and the duality takes a simpler form. This case with $w=1$ is considered in [24].

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